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Generalized algebraic framework for open spin chains

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Abstract. An extension of the Sklyanin algebraic framework for construction of a commuting set of operators is presented. The conditions under which the operators can be interpreted as integrals of motion of an open spin chain with boundary conditions and nearest-neighbour interaction are investigated. An example based on the asymmetric six-vertex solutions of the YBE equation is given.

1. Introduction

At the end of the seventies the quantum inverse scattering method was developed (for a review see, for example, [1]). One of the systems where it was applied was *periodic* spin chain with nearest-neighbour interaction. The algebra, from which the Hamiltonian of this system as well as its integrals of motion were derived, is defined by the relations

$$R_{12}(u_1-u_2)L_1(u_1)L_2(u_2)=L_2(u_2)L_1(u_1)R_{12}(u_1-u_2) \tag{1}$$

where R is a matrix function $R:U\rightarrow End(V_0\otimes V_0)$ satisfying the Yang–Baxter equation (YBE)

$$R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)=R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3). \tag{2}$$

The range U of the 'spectral parameters' u is usually the field of complex or real numbers.

In 1988, Sklyanin proposed a method for constructing solvable models of quantum *open* (i.e. non-periodic) *spin chains* [2]. The method is based on reflection-type algebras given by the relations

$$R_{12}(u_1-u_2)M_1(u_1)R_{12}(u_1+u_2)M_2(u_2)=M_2(u_2)R_{12}(u_1+u_2)M_1(u_1)R_{12}(u_1-u_2)$$

$$R_{12}(u_2-u_1)K_1^{t_1}(u_1)R_{12}(-u_1-u_2-2\eta)K_2^{t_2}(u_2)$$
$$=K_2^{t_2}(u_2)R_{12}(-u_1-u_2-2\eta)K_1^{t_1}(u_1)R_{12}(u_2-u_1).$$

In addition to the YBE, the matrix R was required to satisfy the conditions

$$P_{12}R_{12}(u)P_{12}=R_{12}(u)$$

$$R_{12}^{t_1}(u)=R_{12}^{t_2}(u)$$

$$R_{12}(u)R_{12}(-u)=\rho(u)\mathbf{1}_{12}$$

$$R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta)=\tilde{\rho}(u)\mathbf{1}_{12}$$

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where t_1, t_2 mean the transposition in the first or second pair of indices of $R_{12}(u) = \{R_{i_1 i_2}{}^{j_1 j_2}(u)\}_{i_1,i_2,j_1,j_2=1}^{\dim V_0}$. Under these conditions it was shown that there is a set of mutually commuting elements that can be used for construction of the Hamiltonian for the open spin chain system with the nearest-neighbour interaction and boundary terms. Some of the conditions were later weakened [3] but they still remained rather restrictive. The purpose of this paper is to present a more general construction of open spin chains.

2. The generalized construction

The starting point is the associative algebra generated by elements $M_i^j(u)$, $K_i^j(u)$, $i, j \in \{1, \ldots, d_0 = \dim V_0\}$ satisfying quadratic relations

$$A_{12}(u_1, u_2)M_1(u_1)B_{12}(u_1, u_2)M_2(u_2) = M_2(u_2)C_{12}(u_1, u_2)M_1(u_1)D_{12}(u_1, u_2) \tag{3}$$

$$\tilde{A}_{12}(u_1, u_2)K_1^{t_1}(u_1)\tilde{B}_{12}(u_1, u_2)K_2^{t_2}(u_2) = K_2^{t_2}(u_2)\tilde{C}_{12}(u_1, u_2)K_1^{t_1}(u_1)\tilde{D}_{12}(u_1, u_2) \tag{4}$$

where $A, B, \ldots, \tilde{D}$ are matrix functions $U \times U \to End(V_0 \times V_0)$, i.e. $A_{12}(u_1, u_2) = \{A_{i_1 i_2}^{j_1 j_2}(u_1, u_2)\}_{i_1,i_2,j_1,j_2=1}^{\dim V_0}$ and similarly for $B, C, \ldots, \tilde{D}$. Algebras of this type were investigated in a different context in [4].

Our first task is to identify the conditions for the numerical matrices $A, B, \ldots, \tilde{D}$ that guarantee the existence of a commuting subalgebra which, when represented, will provide us with a commuting set of operators that eventually can be interpreted as the integrals of motion of a quantum system.

Theorem 1. Let $\mathcal{A}$ is the associative algebra generated by elements $M_i^j(u)$, $K_i^j(u)$, relations (3), (4) and

$$M_1(u_1)K_2(u_2) = K_2(u_2)M(u_1). \tag{5}$$

If the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, are related to A, B, C, D by

$$\tilde{A}_{12}(u_1, u_2) = \tilde{a}(u_1, u_2)(A_{12}^{t_1 t_2}(u_1, u_2))^{-1} \tag{6}$$

$$\tilde{B}_{12}(u_1, u_2) = \tilde{b}(u_1, u_2)((B_{12}^{t_1}(u_1, u_2))^{-1})^{t_2} \tag{7}$$

$$\tilde{C}_{12}(u_1, u_2) = \tilde{c}(u_1, u_2)((C_{12}^{t_2}(u_1, u_2))^{-1})^{t_1} \tag{8}$$

$$\tilde{D}_{12}(u_1, u_2) = \tilde{d}(u_1, u_2)(D_{12}^{t_1 t_2}(u_1, u_2))^{-1} \tag{9}$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are scalar functions satisfying

$$\tilde{a}(u_1, u_2)\tilde{b}(u_1, u_2) = \tilde{c}(u_1, u_2)\tilde{d}(u_1, u_2) \tag{10}$$

then the elements $t(u) = K_i^j(u)M_j^i(u) = \mathrm{tr}[K(u)M(u)]$ form a commutative subalgebra of $\mathcal{A}$

$$[t(u_1), t(u_2)] = 0.$$

Remark. Note that there are no restrictions on A, B, C, D.

Proof. This repeats the steps of [2]. Denoting $K_1 \equiv K_1(u_1)$, $K_2 \equiv K_2(u_2)$, $M_1 \equiv M_1(u_1)$, $M_2 \equiv M_2(u_2)$, $A_{12} \equiv A_{12}(u_1, u_2), \ldots, D_{12} \equiv D_{12}(u_1, u_2)$ and using the properties of the trace

$$\mathrm{tr}\, X^t Y^t = \mathrm{tr}\, XY \qquad \mathrm{tr}(XY^t) = \mathrm{tr}(X^t Y)$$

and relations (3)–(8) we get

$$t(u_1)t(u_2) = \mathrm{tr}_1(K_1M_1)\,\mathrm{tr}_2(K_2M_2) = \cdots = \mathrm{tr}_{12}(K_1^{t_1}K_2M_1^{t_1}M_2)$$

$$= \mathrm{tr}_{12}(K_1^{t_1}K_2\tilde{B}_{12}^{t_2}B_{12}^{t_1}M_1^{t_1}M_2)/\tilde{b} = \mathrm{tr}_{12}[(K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_2}(M_1B_{12}M_2)^{t_1}]/\tilde{b}$$

$$= \mathrm{tr}_{12}[(K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_1t_2}\tilde{A}_{12}^{t_1t_2}A_{12}(M_1B_{12}M_2)]/(\tilde{a}\tilde{b})$$

$$= \mathrm{tr}_{12}[(\tilde{A}_{12}K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_1t_2}(A_{12}M_1B_{12}M_2)]/(\tilde{a}\tilde{b})$$

$$= \mathrm{tr}_{12}[(K_2^{t_2}\tilde{C}_{12}K_1^{t_1}\tilde{D}_{12})(M_2C_{12}M_1D_{12})^{t_1t_2}]/(\tilde{c}\tilde{d})$$

$$= \mathrm{tr}_{12}[(K_2^{t_2}\tilde{C}_{12}K_1^{t_1})^{t_1}(M_2C_{12}M_1)^{t_2}]/\tilde{c}$$

$$= \mathrm{tr}_{12}[K_2^{t_2}K_1M_2^{t_2}M_1]$$

$$= \mathrm{tr}_{12}[K_2^{t_2}M_2^{t_2}K_1M_1] = t(u_2)t(u_1).$$

□

The fundamental property that enabled us to construct the operators describing integrals of motion of periodic spin chains was the possibility of defining a coproduct in the algebra (1). The commuting operators could then be expressed in the form

$$\hat{t}(u) = \mathrm{tr}[L_{(N)}(u)L_{(N-1)}(u)\cdots L_{(1)}(u)]$$

where $L_{(j)}$ were matrices of operators acting non-trivially only in the space of the jth spin. Even though it is not possible (in the unbraided categories) to define a coproduct in the algebra $\mathcal{A}$, we can use the algebra for the construction of spin chain operators due to the following covariance property.

Theorem 2. Let $\mathcal{B}$ be the algebra generated by $L(u) = L_i^j(u), N(u) = N_i^j(u)$, $i, j \in \{1, \dots, d_0 = \dim V_0\}$ and the relations

$$A_{12}(u_1, u_2)L_1(u_1)L_2(u_2) = L_2(u_2)L_1(u_1)A_{12}(u_1, u_2) \tag{11}$$

$$D_{12}(u_1, u_2)N_1(u_1)N_2(u_2) = N_2(u_2)N_1(u_1)D_{12}(u_1, u_2) \tag{12}$$

$$N_1(u_1)B_{12}(u_1, u_2)L_2(u_2) = L_2(u_2)B_{12}(u_1, u_2)N_1(u_1) \tag{13}$$

$$L_1(u_1)C_{12}(u_1, u_2)N_2(u_2) = N_2(u_2)C_{12}(u_1, u_2)L_1(u_1). \tag{14}$$

Then:
(i) the algebra $\mathcal{B}$ can be turned into bialgebra by the coproduct

$$\Delta(L_i^j(u)) = L_i^k(u) \otimes L_k^j(u) \qquad \Delta(N_i^j(u)) = N_k^j(u) \otimes N_i^k(u) \tag{15}$$

and counit

$$\epsilon(L_i^j(u)) = \delta_i^j \qquad \epsilon(N_i^j(u)) = \delta_i^j. \tag{16}$$

(ii) The algebra $\mathcal{M}$ generated by the $M_i^j(u)$ and relations (3) is the $\mathcal{B}$-comodule algebra. The coaction on $\mathcal{M}$ is given by

$$\beta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B} \qquad \beta(M_i^j(u)) = M_k^l(u) \otimes L_i^k(u)N_l^j(u) \tag{17}$$

which, with a slight abuse of notation, can be written as $\beta(M) = \tilde{M} = LMN$.

Remark. A similar covariance algebra can be defined for the algebra $\mathcal{K}$ generated by $K_i^j(u)$.

Proof. The check of invariance of the relations (11)–(14) under (15),(16) is straightforward. The invariance of (3) under (17) is proved by

$$A_{12}\tilde{M}_1 B_{12}\tilde{M}_2 = A_{12}L_1M_1(N_1B_{12}L_2)M_2N_2 = (A_{12}L_1L_2)M_1B_{12}N_1M_2N_2$$
$$= L_2L_1(A_{12}M_1B_{12}M_2)N_1N_2 = L_2L_1M_2C_{12}M_1(D_{12}N_1N_2)$$
$$= L_2M_2(L_1C_{12}N_2)M_1N_1D_{12} = (L_2M_2N_2)C_{12}(L_1M_1N_1)D_{12}$$
$$= \tilde{M}_2C_{12}\tilde{M}_1D_{12}.$$

□

(As in the proof of theorem 1, we have omitted the explicit (u_1, u_2) dependence in the above formulae.)

The importance of theorem 2 lies in the fact that it presents the possibility of defining a set of commuting operators composed from the operators acting non-trivially only in the spaces V_i of single spin states. Indeed, if ρ_i are the representation of $\mathcal{B}$ on the spaces V_i, $i = 1, \ldots, N$, then

$$\hat{L}(u) := (\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N) \circ (\Delta^{N-1})(L(u))$$
$$\hat{N}(u) := (\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N) \circ (\Delta^{N-1})(N(u))$$

are operators that represent the algebra $\mathcal{B}$ on $\mathcal{H} \equiv V_1 \otimes V_2 \otimes \cdots \otimes V_N$ that is the Hilbert space of the system of N spins. The operators $\hat{L}(u)$, $\hat{N}(u)$ can be written as

$$\hat{L}(u) = \hat{L}_{(N)}(u)\hat{L}_{(N-1)}(u) \cdots \hat{L}_{(1)}(u) \tag{18}$$

$$\hat{N}(u) = \hat{N}_{(1)}(u)\hat{N}_{(2)}(u) \cdots \hat{N}_{(N)}(u) \tag{19}$$

where

$$\hat{L}_{(j)}(u) = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \rho_j(L(u)) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$
$$\hat{N}_{(j)}(u) = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \rho_j(N(u)) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

The representations of $\mathcal{B}$ on V_i such that $\dim V_i = \dim V_0$, $i = 1, 2, \ldots, N$ follow from theorem 3.

Theorem 3. Let there be $\alpha, \delta \in U$ such that the matrices A, B, C, D satisfy the equations

$$A_{12}(u_1, u_2)A_{13}(u_1, \alpha)A_{23}(u_2, \alpha) = A_{23}(u_2, \alpha)A_{13}(u_1, \alpha)A_{12}(u_1, u_2) \tag{20}$$

$$D_{12}(u_1, u_2)D_{13}(u_1, \delta)D_{23}(u_2, \delta) = D_{23}(u_2, \delta)D_{13}(u_1, \delta)D_{12}(u_1, u_2) \tag{21}$$

$$D_{13}(u_1, \delta)B_{12}(u_1, u_2)A_{23}(u_2, \alpha) = A_{23}(u_2, \alpha)B_{12}(u_1, u_2)D_{13}(u_1, \delta) \tag{22}$$

$$A_{13}(u_1, \alpha)C_{12}(u_1, u_2)D_{23}(u_2, \delta) = D_{23}(u_2, \delta)C_{12}(u_1, u_2)A_{13}(u_1, \alpha) \tag{23}$$

for all $u_1, u_2 \in U$. (Note the unusual order of indices in (22), (23).)

Then the multiplicative map $\rho_{\alpha\delta} : \mathcal{B} \to End(V_0)$

$$[\rho_{\alpha\delta}(L^j_k(u))]^n_m = A^{jn}_{km}(u, \alpha) \tag{24}$$

$$[\rho_{\alpha\delta}(N^j_k(u))]^n_m = D^{jn}_{km}(u, \delta) \tag{25}$$

is a representation of the algebra $\mathcal{B}$ on a space V such that $\dim V = \dim V_0$.

Proof. This is by direct check of relations (11)–(14) by means of (20)–(23).

Remark. Note that full Yang–Baxter-type equations are not required in the theorem. It is sufficient if they are satisfied for single $(\alpha, \delta) \in U \times U$.

If we find a representation σ of $\mathcal{A}$ on $\mathcal{H}$, then due to theorems 1 and 2 we get the set of commuting operators on $\mathcal{H}$

$$\hat{t}(u) := \mathrm{Tr}[\sigma(K(u))\hat{L}(u)\sigma(M(u))\hat{N}(u)]$$

from which the Hamiltonians can be extracted. The simplest possibility is given by one-dimensional representation of $\mathcal{A}$. Assuming that there are numerical matrices $m(u), k(u) \in End(V_0)$ that satisfy relations (3), (4), we can choose

$$\sigma(M_i^j(u)) = m_i^j(u)\mathbb{1}_{\mathcal{H}} \qquad \sigma(K_i^j(u)) = k_i^j(u)\mathbb{1}_{\mathcal{H}}.$$

Then

$$\hat{t}(u) = \mathrm{tr}[k(u)\hat{L}_{(N)}(u)\dots\hat{L}_{(1)}(u)m(u)\hat{N}_{(1)}(u)\dots\hat{N}_{(N)}(u)] \tag{26}$$

where the operator matrices $L_{(k)}$ and $N_{(k)}$ act non-trivially only in the kth factor of the space $\mathcal{H} = V_0 \otimes V_0 \otimes \cdots \otimes V_0$.

The last goal we want to achieve is finding the Hamiltonian H of the open chain system with nearest-neighbour interaction and boundary terms. For that we need the so-called regularity conditions.

Theorem 4. Let there be a one-dimensional representation of the algebra $\mathcal{A}$ by the numerical matrices $m(u)$, $k(u)$ and the representation of $\mathcal{B}$ on V_i be $\rho_i = \rho_{\alpha,\delta}$ for all $i \in \{1, \dots, N\}$.

If there is $u_0 \in U$ such that

$$A_{12}(u_0, \alpha) = \kappa P_{12} \qquad D_{12}(u_0, \delta) = \lambda P_{12} \qquad m(u_0) = \mu\mathbb{1} \qquad \mathrm{tr}[k(u_0)] \neq 0 \tag{27}$$

where κ, λ, μ are constants and P is the permutation matrix, then

$$\hat{t}(u_0) = \mu(\kappa\lambda)^N \,\mathrm{tr}[k(u_0)]\mathbb{1}_{\mathcal{H}} \tag{28}$$

and

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}u}(u_0) = (\kappa\lambda)^N\mu\left[H\,\mathrm{tr}[k(u_0)] + \mathrm{tr}\left(\frac{\mathrm{d}k}{\mathrm{d}u}(u_0)\right)\mathbb{1}_{\mathcal{H}}\right] \tag{29}$$

where the Hamiltonian H is a sum of operators acting non-trivially only on V_1, $V_j \otimes V_{j+1}$, $j = 1, \dots, N-1$ and V_N.

$$H = \sum_{n=1}^{N-1} H_{n,n+1} + \mu^{-1}\frac{\mathrm{d}m_{(1)}}{\mathrm{d}u}(u_0) + [\mathrm{tr}\;k(u_0)]^{-1}\,\mathrm{tr}_0[k_0(u_0)H_{N,0}] \tag{30}$$

$$H_{n,n+1} = \lambda^{-1}\frac{\mathrm{d}D_{n,n+1}}{\mathrm{d}u}(u_0, \delta)P_{n,n+1} + \kappa^{-1}P_{n,n+1}\frac{\mathrm{d}A_{n,n+1}}{\mathrm{d}u}(u_0, \alpha). \tag{31}$$

Proof. From (18), (19) and (24)–(27) we get

$$\hat{L}(u) = \kappa^N P_{0N}P_{0n-1}, \dots, P_{01} \qquad \hat{N}(u) = \lambda^N P_{01}P_{02}, \dots, P_{0N} \tag{32}$$

from which (28) follows immediately. Similarly, (29)–(31) are obtained by differentiating (26), using (32) and the following identity on $End(V_0^{\otimes N+1})$:

$$P_{0,n+1}X_{0n} = X_{n+1,n}P_{0,n+1}.$$

We can see that the *open spin chains with the nearest-neighbour interaction can be constructed from rather general quadratic algebras* defined by matrix functions A, B, C, D of two variables that:
(i) satisfy the equations (20)–(23);
(ii) admit one-dimensional representation of $\mathcal{A}$, i.e. numerical matrices $m(u), k(u)$ that satisfy (3), (4), where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are given by (9), (10);
(iii) satisfy the regularity condition (27) for A, D and $m(u)$, $k(u)$.

3. Example: the XXZ model in a magnetic field

The above theory suggests the following procedure for construction of the open spin chain models:
(i) Take two solutions A, D of the YBE (of the same dimension).
(ii) Find the matrices B, C that satisfy (22), (23).
(iii) Solve equations (3), (4) for numerical matrices $m(u), k(u)$.
(iv) Check the regularity conditions.
(v) Evaluate the Hamiltonian (30), (31).

In this section we are going to apply this procedure to spaces with dim = 2, i.e. spin-$\frac{1}{2}$ chains and asymmetric six-vertex matrices

$$A(u,v) = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & A_5 & 0 \\ 0 & A_6 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \qquad D(u,v) = \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & D_5 & 0 \\ 0 & D_6 & D_3 & 0 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \tag{33}$$

where $A_i = A_i(u,v)$, $D_i = D_i(u,v)$, $i = 1, 2, \ldots, 6$.

Inserting equation (33) into (22) we get a system of linear homogeneous equations for the elements of the matrix B. Solving it by the standard method we find that if $A_1 \neq A_3$, $A_2 \neq A_4$, $D_1 \neq D_3$, $D_2 \neq D_4$, then there is a non-zero solution of (22) if and only if

$$\frac{\tilde{A}_F}{\tilde{A}_1\tilde{A}_2} = \frac{\bar{D}_F}{\bar{D}_3\bar{D}_4} \qquad \frac{\bar{D}_F}{\bar{D}_1\bar{D}_2} = \frac{\tilde{A}_F}{\tilde{A}_3\tilde{A}_4} \tag{34}$$

where $\tilde{A}_i = A_i(v,\alpha)$, $\bar{D}_i = D_i(u,\delta)$ and

$$\tilde{A}_F = \tilde{A}_1\tilde{A}_4 + \tilde{A}_2\tilde{A}_3 - \tilde{A}_5\tilde{A}_6 \tag{35}$$

$$\bar{D}_F = \bar{D}_1\bar{D}_4 + \bar{D}_2\bar{D}_3 - \bar{D}_5\bar{D}_6. \tag{36}$$

The solution is

$$B = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & B_5 & 0 \\ 0 & B_6 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix} \tag{37}$$

$$B_1 = \tilde{A}_1\bar{D}_1 - \tilde{A}_3\bar{D}_3 \qquad B_4 = \tilde{A}_4\bar{D}_4 - \tilde{A}_2\bar{D}_2 \tag{38}$$

$$B_2 = \tilde{A}_3\bar{D}_4 + (\tilde{A}_5\tilde{A}_6 - \tilde{A}_2\tilde{A}_3)\bar{D}_2/\tilde{A}_4 \tag{39}$$

$$B_3 = \tilde{A}_4\bar{D}_3 + (\bar{D}_5\bar{D}_6 - \bar{D}_2\bar{D}_3)\tilde{A}_2/\bar{D}_4 \tag{40}$$

$$B_5 = \tilde{A}_5\bar{D}_6 \qquad B_6 = \tilde{A}_6\bar{D}_5. \tag{41}$$

It is unique up to a scalar factor.

Note that the conditions (34) imply that either both the matrices A and D are free-fermionic, i.e. $A_F = 0$, $D_F = 0$, or non-free-fermionic, i.e. $A_F \neq 0$, $D_F \neq 0$.

We shall deal with the latter case because for the free-fermionic solutions usually $\mathrm{tr}[k(u_0)] = 0$ [7] and the formula (31) cannot be applied. Let A and D be non-symmetric solutions of the YBE [5]

$$A(u,v)=\begin{pmatrix} \frac{\varphi(u)}{\varphi(v)}\sin(u-v+\eta) & 0 & 0 & 0 \\ 0 & \frac{\sin(u-v)}{\varphi(u)\varphi(v)} & \sin(\eta) & 0 \\ 0 & \sin(\eta) & \varphi(u)\varphi(v)\sin(u-v) & 0 \\ 0 & 0 & 0 & \frac{\varphi(v)}{\varphi(u)}\sin(u-v+\eta) \end{pmatrix} \tag{42}$$

where φ is an arbitrary function and $D(u, v)$ is obtained from $A(u, v)$ by $\varphi \to \varphi'$.

The condition (34) reads

$$\varphi'(\delta)^2\varphi(\alpha)^2 = 1 \tag{43}$$

and inserting A, D into (37)–(41) we get

$$B(u,v)=\begin{pmatrix} \varphi'(u)\varphi(v)\sin(\omega+\eta) & 0 & 0 & 0 \\ 0 & \frac{\varphi(v)}{\varphi'(u)}\sin(\omega) & p\sin(\eta) & 0 \\ 0 & p\sin(\eta) & \frac{\varphi'(u)}{\varphi(v)}\sin(\omega) & 0 \\ 0 & 0 & 0 & \frac{\sin(\omega+\eta)}{\varphi(v)\varphi'(u)} \end{pmatrix} \tag{44}$$

where $\omega = u + v - \alpha - \delta$ and $p = \varphi'(\delta)\varphi(\alpha)$.

It is easy to see that the solution of (23) can be derived from that of (22) by transformation $A \leftrightarrow D$, $\alpha \leftrightarrow \delta$ or alternatively by $u \leftrightarrow v$, $2 \leftrightarrow 1$ so that

$$C(u, v) = PB(v, u)P \tag{45}$$

up to a scalar factor.

The next step is solving the relations (3), (4) in terms of numerical matrices $m(u)$, $k(u)$. If we are looking for the diagonal solution

$$m(u) = \begin{pmatrix} x(u) & 0 \\ 0 & t(u) \end{pmatrix}$$

then equation (3) yields (for a suitable normalization of the matrices A, B, C, D) only one equation for $y(u) = x(u)/t(u)$, namely

$$\left(\frac{x(u)x(v)}{\varphi(u)\varphi'(u)\varphi(v)\varphi'(v)} - t(u)t(v)\right)\sin(u-v)$$
$$= p\left(\frac{x(u)t(v)}{\varphi(u)\varphi'(u)} - \frac{x(v)t(u)}{\varphi(v)\varphi'(v)}\right)\sin(u+v-\alpha-\delta). \tag{46}$$

It can be transformed to the well known functional equation

$$(Y(u)Y(v) - 1)\sin(u - v) = (Y(u) - Y(v))\sin(u + v - \alpha - \delta) \tag{47}$$

by the transformation

$$Y(u-\alpha/2+\delta/2)=\frac{p\,x(u)}{t(u)\varphi(u)\varphi'(u)}. \tag{48}$$

The diagonal solution of (3) is then

$$m(u)=\begin{pmatrix}\varphi(u)\varphi'(u)\sin(\xi_-+u-\alpha/2-\delta/2) & 0\\ 0 & p\,\sin(\xi_-u+\alpha/2+\delta/2)\end{pmatrix}. \tag{49}$$

The equation (4) can be obtained from (3) by $\varphi\to 1/\varphi$, $\varphi'\to 1/\varphi'$, $M(u)\to K^t(u)$ and changing the arguments of sin by $u\to -u-\eta+\alpha+\delta$, $v\to -v-\eta+\alpha+\delta$. Applying these transformations we find that

$$k(u)=\begin{pmatrix}p\,\sin(\xi_+-u+\alpha/2+\delta/2-\eta) & 0\\ 0 & \varphi(u)\varphi'(u)\sin(\xi_++u-\alpha/2-\delta/2+\eta)\end{pmatrix} \tag{50}$$

(ξ_- and ξ_+ in (49) and (50) are arbitrary constants).

The regularity conditions (27) are satisfied if

$$u_0=\alpha=\delta$$

and the nearest-neighbour interaction Hamiltonian derived from (30), (31) is

$$H=(2\sin\eta)^{-1}\sum_{n=1}^{N-1}\left\{\cosh\theta(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y)+\mathrm{i}\sinh\theta(\sigma_n^x\sigma_{n+1}^y-\sigma_n^y\sigma_{n+1}^x)\right.$$
$$\left.+\cos\eta\sigma_n^z\sigma_{n+1}^z\right\}+h\sum_{n=1}^{N}\sigma_n^z+\sigma_1^z\cot\xi_-+\sigma_N^z\cot\xi_+$$

where

$$\exp\theta=\varphi(u_0)^2=\varphi'(u_0)^{-2}\qquad h=\left.\frac{\mathrm{d}\log\varphi(u)\varphi'(u)}{\mathrm{d}u}\right|_{u=u_0}.$$

This Hamiltonian is an open version of the XXZ model in the homogeneous magnetic field h and is an extension of the models presented in [2, 3, 8].

The non-diagonal matrices $m(u)$, $k(u)$ can be obtained only when $\varphi'(u)\varphi(u)=$ constant. In that case we obtain a Hamiltonian with the boundary terms proportional to σ^x and σ^y, like in [9], but the external homogeneous magnetic field h vanishes.

4. Conclusions

The algebraic framework for the construction of integrable models can be extended to quadratic algebras whose 'structure coefficients' are given by a *pair of solutions* A and D of the spectral-dependant YBE.

No symmetries of the solutions are required but a certain compatibility between them must be satisfied in order that the algebras may have convenient covariance properties. In case of six-vertex models the compatibility means that either free-fermion or non-free-fermion solutions can be used.

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